Indian Statistical Institute, Bangalore
B. Math. Second Year
First Semester - Analysis III

Semestral Exam Date : Nov 07, 2014

Section I: Answer any four, each question carries 6 marks, total marks: 24

1. Let (f_n) be a sequence of functions on a metric space E such that (f_n) converges uniformly to a function f on E and x be a limit point of E. If $\lim_{t\to x} f_n(t) = A_n$, prove that (A_n) converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

2. Let (f_n) be a uniformly converging sequence of bounded functions and $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that $(g \odot f_n)$ converges uniformly. Does the result hold if f_n are not bounded? Justify your answer.

3. Do flips and primitive maps on $\mathbb{R}^d$ have potential? Justify your answer.

4. (a) If r and R are smoothly equivalent surfaces and equivalence is given by $G = (U, V)$, prove that $\frac{\partial R}{\partial s} \times \frac{\partial R}{\partial t} = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \frac{\partial(U,V)}{\partial(s,t)}$.

   (b) Find the fundamental vector product for hemisphere *(Marks: 2)*.

5. Let $R = [a, b] \times [c, d]$ and Γ be its positively oriented boundary. If u, v are scalar valued functions on R with continuous second order partial derivatives, prove that $\int_\Gamma (v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x})dx + (u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y})dy = 2\int\int_R (u\frac{\partial^2 v}{\partial x \partial y} - v\frac{\partial^2 u}{\partial x \partial y})dxdy$.

6. Let K be a compact subset of $\mathbb{R}^d$ and $\{V_i\}$ be an open cover of K. Prove that there are $f_1, \cdots, f_n \in C_c(\mathbb{R}^d)$ such that $0 \leq f_i \leq 1$ and $\sum f_i = 1$ on K.

Section II: Answer any two, each question carries 13 marks, total marks: 26

1. (a) Let (f_n) be a sequence of pointwise bounded functions on a countable set E. Prove that (f_n) has a subsequence that converges pointwise on E.

   (b) Let (f_n) be a sequence of continuous functions on $[0, 1]$. If (f_n) converges uniformly on $[0, 1]$, prove or disprove $\lim_{n\to\infty}[\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f_n] = 0$ *(Marks: 7)*.

2. (a) Let $\alpha\colon [a, b] \to \mathbb{R}^2$ be a smooth closed curve and $P\colon \alpha([a, b]) \to \mathbb{R}$ be a continuous function. Prove that $|\int P dx| \leq \frac{1}{2}(\text{Max}P - \text{Min}P)\Lambda(\alpha)$ *(Marks: 7)*.

   (b) If $Q = \{(x_1, x_2, x_3) \in [0, \infty)^3 \mid x_1+x_2+x_3 \leq 1\}$ and r_1, r_2, r_3 are nonnegative integers, prove that $\int_Q x_1^{r_1} x_2^{r_2} x_3^{r_3} dx = \frac{r_1!r_2!r_3!}{(3+r_1+r_2+r_3)!}$. [P.T.O]

3. (a) Let C be the triangle with vertices $(0,0),(1,0)$ and $(1,3)$ with clock-wise orientation. Find $\int_C \sqrt{1+x^5}dx + 2xydy$ and $\int_C p(x)dx + q(y)dy$ where p and q are real polynomials.

   (b) Let T be an open set in $\mathbb{R}^2$ whose boundary is a piecewise smooth Jordan curve Γ with positive orientation and r be a $\mathbb{R}^3$-valued function defined on an open set containing $T \cup \Gamma$ such that r defines a smooth surface $S = r(T)$ and r has continuous 2nd order partial derivatives. If P is a continuously differentiable function defined on $S = r(T)$, prove that $\int_{r(\Gamma)} Pdx = \int\int_S \frac{\partial P}{\partial z} dz \wedge dx - \frac{\partial P}{\partial y} dx \wedge dy$ *(Marks: 7)*.